Problem Set 10 – Relativity (`GOI36A`)

1 Conservation of the Stress-Energy Tensor

1.) First, we note that $y^\mu = x^\mu + \xi^\mu(x)$, which we can invert to write $x^\mu = y^\mu - \xi^\mu(x)$. The partial derivative of x with respect to y is then given by

$$\begin{aligned}
\frac{\partial x^\rho}{\partial y^\mu} &= \frac{\partial(y^\rho - \xi^\rho(x))}{\partial y^\mu} = \delta^\rho_\mu - \frac{\partial \xi^\rho(x)}{\partial y^\mu} \\
&= \delta^\rho_\mu - \frac{\partial \xi^\rho(x)}{\partial x^\nu}\frac{\partial x^\nu}{\partial y^\mu} \\
&= \delta^\rho_\mu - \frac{\partial \xi^\rho(x)}{\partial x^\nu}\left(\delta^\nu_\mu - \frac{\partial \xi^\nu(x)}{\partial y^\mu}\right) .
\end{aligned} \tag{1.1}$$

Notice that this is a recursive relationship. We could in principle keep iterating and expand to arbitrary order in the small parameter ξ, but since ξ is infinitesimal we will only need to keep the leading order piece. Thus, we conclude that

$$\frac{\partial x^\rho}{\partial y^\mu} = \delta^\rho_\mu - \frac{\partial \xi^\rho(x)}{\partial x^\nu}\delta^\nu_\mu + \mathcal{O}(\xi^2) = \delta^\rho_\mu - \partial_\mu \xi^\rho(x) + \mathcal{O}(\xi^2) , \tag{1.2}$$

where ∂_μ is short-hand for $\frac{\partial}{\partial x^\mu}$. Using this result, the transformation of the metric is given by

$$\begin{aligned}
\tilde{g}_{\mu\nu}(y) &= \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\sigma} g_{\rho\sigma}(x) \\
&= \left(\delta^\rho_\mu - \partial_\mu \xi^\rho(x) + \mathcal{O}(\xi^2)\right)\left(\delta^\sigma_\nu - \partial_\nu \xi^\sigma(x) + \mathcal{O}(\xi^2)\right) g_{\rho\sigma}(x) \\
&= g_{\rho\sigma}(x)\delta^\rho_\mu\delta^\sigma_\nu - g_{\rho\sigma}(x)\delta^\sigma_\nu\partial_\mu\xi^\rho(x) - g_{\rho\sigma}(x)\delta^\rho_\mu\partial_\nu\xi^\sigma(x) + \mathcal{O}(\xi^2) \\
&= g_{\mu\nu}(x) - g_{\nu\rho}(x)\partial_\mu\xi^\rho(x) - g_{\mu\sigma}(x)\partial_\nu\xi^\sigma(x) + \mathcal{O}(\xi^2) .
\end{aligned} \tag{1.3}$$

2.) First, for an example, consider a one-dimensional function f for which we know the value of f and its derivatives at a particular point x. If ϵ is some small number, than we can compute $f(x+\epsilon)$ via a Taylor expansion of the form

$$f(x+\epsilon) = f(x) + \epsilon f'(x) + \frac{1}{2!}\epsilon^2 f''(x) + \ldots \tag{1.4}$$

Now, we note that since $y^\mu = x^\mu + \xi^\mu$ and ξ^μ is very small, we should be able to do a very similar Taylor series in order to express $\tilde{g}_{\mu\nu}(y)$ in terms of $\tilde{g}_{\mu\nu}(x)$ and its derivatives. This particular, this Taylor expansion looks like

$$\tilde{g}_{\mu\nu}(y) = \tilde{g}_{\mu\nu}(x) + \xi^\rho(x)\partial_\rho\tilde{g}_{\mu\nu}(x) + \frac{1}{2!}\xi^\rho(x)\xi^\sigma(x)\partial_\rho\partial_\sigma\tilde{g}_{\mu\nu}(x) + \ldots , \tag{1.5}$$

where the dots indicate higher-order derivatives. Ignoring all $\mathcal{O}(\xi^2)$ terms and keeping just the zeroth and first order pieces, this is simply

$$\tilde{g}_{\mu\nu}(y) = \tilde{g}_{\mu\nu}(x) + \xi^\rho(x)\partial_\rho\tilde{g}_{\mu\nu}(x) + \mathcal{O}(\xi^2) . \tag{1.6}$$

Then, since once again ξ is very small, it should be true that $\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathcal{O}(\xi)$, i.e. that the transformed metric is very close to the original metric when evaluated at the same spacetime coordinate. This in turn means that

$$\xi^\rho(x)\partial_\rho\tilde{g}_{\mu\nu}(x) = \xi^\rho(x)\partial_\rho(g_{\mu\nu}(x) + \mathcal{O}(\xi)) = \xi^\rho\partial_\rho(x)g_{\mu\nu}(x) + \mathcal{O}(\xi^2)\ . \tag{1.7}$$

Combining (1.6) and (1.7), we find the desired result:

$$\tilde{g}_{\mu\nu}(y) = \tilde{g}_{\mu\nu}(x) + \xi^\rho(x)\partial_\rho g_{\mu\nu}(x) + \mathcal{O}(\xi^2)\ . \tag{1.8}$$

3.) Our results from problems 1 and 2 give us two different expressions for $\tilde{g}_{\mu\nu}(y)$. Setting them equal, we find that

$$\tilde{g}_{\mu\nu}(x) + \xi^\rho(x)\partial_\rho g_{\mu\nu}(x) + \mathcal{O}(\xi^2) = g_{\mu\nu}(x) - g_{\nu\rho}(x)\partial_\mu\xi^\rho(x) - g_{\mu\sigma}(x)\partial_\nu\xi^\sigma(x) + \mathcal{O}(\xi^2)\ . \tag{1.9}$$

Bringing $g_{\mu\nu}(x)$ to the left-hand side, moving all ξ-dependent pieces to the right-hand side, and dropping all $\mathcal{O}(\xi^2)$ terms, we find that

$$\tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -\xi^\rho(x)\partial_\rho g_{\mu\nu}(x) - g_{\nu\rho}(x)\partial_\mu\xi^\rho(x) - g_{\mu\sigma}(x)\partial_\nu\xi^\sigma(x). \tag{1.10}$$

At this point, everything is evaluated at the same spacetime coordinate x^μ, and so we will drop this dependence and keep it implicit. Defining $\delta g_{\mu\nu} \equiv \tilde{g}_{\mu\nu} - g_{\mu\nu}$, this relation becomes

$$\delta g_{\mu\nu} = -\xi^\rho\partial_\rho g_{\mu\nu} - g_{\nu\rho}\partial_\mu\xi^\rho - g_{\mu\sigma}\partial_\nu\xi^\sigma. \tag{1.11}$$

This is a nice simple expression, but we're not quite done yet. We want to use integration by parts in order rewrite this expression in such a way that derivatives act explicitly on the metric. That is, we can rewrite the second term in the right-hand side via

$$g_{\nu\rho}\partial_\mu\xi^\rho = \partial_\mu(g_{\nu\rho}\xi^\rho) - \xi^\rho\partial_\mu g_{\nu\rho} = \partial_\mu\xi_\nu - \xi^\rho\partial_\mu g_{\nu\rho}\ . \tag{1.12}$$

A similar expression holds for the third term. The end result of this rewriting is that, after relabelling $\sigma \to \rho$, we find the desired expression:

$$\delta g_{\mu\nu} = -\xi^\rho\partial_\rho g_{\mu\nu} - \partial_\mu\xi_\nu + \xi^\rho\partial_\mu g_{\nu\rho} - \partial_\nu\xi_\mu + \xi^\rho\partial_\nu g_{\mu\rho}\ . \tag{1.13}$$

4.) Using the definition of the covariant derivative, and then expanding out all Christoffel symbols explicitly, we compute that

$$\begin{aligned}
-\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu &= -(\partial_\mu\xi_\nu - \Gamma^\rho_{\mu\nu}\xi_\rho) - (\partial_\nu\xi_\mu - \Gamma^\rho_{\nu\mu}\xi_\rho) \\
&= -\partial_\mu\xi_\nu - \partial_\nu\xi_\mu + 2\Gamma^\rho_{\mu\nu}\xi_\rho \\
&= -\partial_\mu\xi_\nu - \partial_\nu\xi_\mu + g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\xi_\rho \\
&= -\partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \xi^\sigma(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\ .
\end{aligned} \tag{1.14}$$

At this point, if we relabel the dummy index $\sigma \to \rho$, you should be able to compare this to (1.13) and see that they are exactly equal. Therefore, the variation in the metric is given very succinctly by

$$\delta g_{\mu\nu} = -\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu\ . \tag{1.15}$$

Then, if we recall that (see earlier problem sets for a discussion of this) that $\delta g_{\mu\nu}$ and $\delta g^{\mu\nu}$ are related by $\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$, we find the remaining thing we needed to prove:

$$\delta g^{\mu\nu} = \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu\ . \tag{1.16}$$

5.) The central idea behind general relativity is that physical fields and objects propagate on a spacetime manifold, the geometry of which is determined by the details of the matter fields. The underlying spacetime manifold has no preferred set of coordinates; we simply pick coordinates on the manifold (thus specifying a metric tensor) in a way that is convenient for us. The underlying physics of how objects propagate and interact on this manifold should not care about how we pick these coordinates! This is the concept of general covariance.

So, this means that we must also choose matter content that is compatible with general covariance. In particular, this means that the matter action S_M must be invariant under coordinate transformations. If it were not, this would mean that scalar quantities (like ${T_\mu}^\mu$) coming from the stress-tensor of this action would vary depending on my coordinate frame. This in turn, via the Einstein equation, would mean that concepts like the curvature of the spacetime would be coordinate dependent, in stark contrast to the general covariance principle. Thus S_M must be invariant under coordinate transformations.

6.) Under a coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, we know from problem 4 that the corresponding variation in the inverse metric is $\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu$. Thus, the variation of the matter action under this coordinate transformation is

$$\delta S_M = -\frac{1}{2}\int d^4x\,\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2}\int d^4x\,\sqrt{-g}\,T_{\mu\nu}(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu)\ . \tag{1.17}$$

Since the stress-energy tensor is symmetric, we can rewrite this as

$$\delta S_M = -\int d^4x\,\sqrt{-g}\,T_{\mu\nu}\nabla^\mu\xi^\nu = -\int d^4x\,\sqrt{-g}\,T^{\mu\nu}\nabla_\mu\xi_\nu\ . \tag{1.18}$$

But, as argued in problem 5, S_M should be invariant under coordinate transformations. So, its variation should be zero! Therefore, we find that

$$\int d^4x\,\sqrt{-g}\,T^{\mu\nu}\nabla_\mu\xi_\nu = 0\ . \tag{1.19}$$

7.) Integration by parts tells us that the above integral can be rewritten as

$$\int d^4x\,\sqrt{-g}\,T^{\mu\nu}\nabla_\mu\xi_\nu = \int d^4x\,\sqrt{-g}\,[\nabla_\mu\,(T^{\mu\nu}\xi_\nu) - (\nabla_\mu T^{\mu\nu})\xi_\nu]\ . \tag{1.20}$$

Then, by Stokes' theorem, the first term on the right-hand side of this expression is a total derivative and thus can be turned into an integral of $n_\mu T^{\mu\nu}\xi_\nu$ on the boundary of the spacetime manifold. We will assume that we can neglect such a boundary term, i.e. that $T_{\mu\nu}$ falls off sufficiently quickly as we move to the boundary such that all boundary terms are subleading to the bulk terms. This leaves us with

$$\int d^4x\,\sqrt{-g}\,T^{\mu\nu}\nabla_\mu\xi_\nu = \int d^4x\,\sqrt{-g}\,(\nabla_\mu T^{\mu\nu})\xi_\nu\ . \tag{1.21}$$

Since the left-hand side of this expression must vanish, the right-hand side must as well.

8.) We will show that $\nabla_\mu T^{\mu\nu} = 0$ via a proof by contradiction. So, assume that there exists some region $\mathcal{R}$ in our spacetime on which $\nabla_\mu T^{\mu\nu} \neq 0$. Since it is non-zero everywhere on $\mathcal{R}$, it must also have a positive-definite or negative-definite sign everywhere on $\mathcal{R}$. Then, since ξ^μ is completely arbitrary, we could very carefully choose ξ_μ such that it is a small constant c_μ on this region $\mathcal{R}$, but zero everywhere else. Then, we would find that

$$\int d^4x\,\sqrt{-g}\,(\nabla_\mu T^{\mu\nu})\xi_\nu = c_\nu \int_{\mathcal{R}} d^4x\,\sqrt{-g}\,\nabla_\mu T^{\mu\nu}\ . \tag{1.22}$$

Since the integrand is either positive-definite or negative-definite on $\mathcal{R}$, the integral itself must also either be positive-definite or negative-definite, and thus the integral cannot vanish. This is a contradiction, since we established in problem 7 that the integral must vanish.

Thus, the only way for the integral to vanish for arbitrary ξ_μ is for $\nabla_\mu T^{\mu\nu} = 0$ *everywhere* on the spacetime manifold.

9.) The Einstein equation tells us that

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ . \tag{1.23}$$

Raising indices and acting with a covariant derivative on this equation, we find that

$$\nabla_\mu R^{\mu\nu} - \frac{1}{2} \nabla^\nu R = 8\pi G \nabla_\mu T^{\mu\nu} = 0 \ , \tag{1.24}$$

where the right-hand side is zero via our result from the previous problem. Thus we immediately find that $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$.

2 Orbits around Reissner-Nordström Black Holes

10.) There are four Killing vectors on this background. One is the usual time-like vector $K = \partial_t$ that is a Killing vector of any stationary metric, such as the Reissner-Nordström metric. The remaining three, which we will denote by R, S, and T are the three Killing vectors of S^2 that correspond to rotations of the two-sphere, since the only place the angular coordinates appear in the metric components are in the $d\Omega^2$ term. K leads to conservation of energy, while R, S, and T lead to conservation of angular momentum.

11.) The fact that our metric is invariant under arbitrary $SO(3)$ rotations of the two-sphere means that we can always choose our coordinates such that geodesic motion is in the $\theta = \pi/2$ plane. That is, the object has a conserved angular momentum vector, and we are just rotating our coordinate axes such that this vector points in the z-direction. This requires doing x- and y-rotations, which are isometries that correspond to two of the Killing vectors S and T.

12.) We have not yet said anything about $K = \partial_t$ and $R = \partial_\phi$. These are both Killing vectors; they will correspond to the time-translation and ϕ-rotation invariance of our system that we have not yet used. The corresponding quantities E and L are conserved, since $V_\mu dx^\mu/d\lambda$ is conserved along a geodesic for any Killing vector V. These conserved quantities are energy and angular momentum, respectively. Writing them out explicitly, they are

$$E = \Delta \frac{dt}{d\lambda} \ , \quad L = r^2 \frac{d\phi}{d\lambda} \ . \tag{2.1}$$

Note that in the above expressions we used the fact that $\theta = \frac{\pi}{2}$ is fixed.

13.) ϵ computes the negation of the norm of the vector $U^\mu = \frac{dx^\mu}{d\lambda}$. Massive particles should travel on time-like geodesics, in order to obey causality, so U is a time-like vector with negative norm, and thus ϵ should be positive. Normalizing this to one, this means that $\epsilon = +1$ for massive particles. Massless particles travel on null geodesics, which corresponds to U having zero norm, and so ϵ is zero for massless particles. Finally, $\epsilon = -1$ corresponds to U being a space-like vector, so we would have to have be describing some space-like observer that moves outside of the light-cone.

14.) In full generality, when we plug in the components of $g_{\mu\nu}$, we find that

$$\epsilon = \Delta\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{\Delta}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\theta}{d\lambda}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 \ . \tag{2.2}$$

We fixed our coordinates such that θ is fixed to be $\pi/2$, and so $d\theta/d\lambda = 0$. We are left with the desired expression:

$$\epsilon = \Delta\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{\Delta}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 \ . \tag{2.3}$$

15.) From problem 12, we know that

$$\frac{dt}{d\lambda} = \frac{E}{\Delta} \ , \quad \frac{d\phi}{d\lambda} = \frac{L}{r^2} \ . \tag{2.4}$$

Plugging these into the previous problem result, we find that

$$\epsilon = \frac{E^2}{\Delta} - \frac{1}{\Delta}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2} \ , \tag{2.5}$$

which we can rearrange terms to write as

$$\left(\frac{dr}{d\lambda}\right)^2 + \epsilon\Delta + \frac{\Delta L^2}{r^2} = E^2 \ . \tag{2.6}$$

If we expand Δ our explicitly and divide by two, this becomes

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\epsilon\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) + \frac{L^2}{2r^2}\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) = \frac{1}{2}E^2 \ . \tag{2.7}$$

From here, it should be clear that this can be rewritten as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E} \ , \tag{2.8}$$

with

$$V(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2 + \epsilon GQ^2}{2r^2} - \frac{GML^2}{r^3} + \frac{GL^2Q^2}{2r^4} \ , \tag{2.9}$$

$$\mathcal{E} = \frac{1}{2}E^2 \ . \tag{2.10}$$

16.) If we have an object moving classically in a one-dimensional potential $V(r)$, then at any point where $V'(r) = 0$, there is a turning point at which the object can sit without moving. This turning point may be stable or unstable, depending on $V''(r)$, but nonetheless a one-dimensional object has stationary points only when $V'(r) = 0$. Such stationary points for our geodesic would mean that r is not changing, while t and ϕ can still both change along the geodesic. Thus, these stationary points are circular orbits of fixed radius. The condition for such circular orbits for our potential is

$$\epsilon GMr^3 - (L^2 + \epsilon GQ^2)r^2 + 3GML^2r - 2GL^2Q^2 = 0 \ . \tag{2.11}$$

17.) For a massless particle we can set $\epsilon = 0$, and so the circular orbit condition is

$$L^2r^2 - 3GML^2r + 2GL^2Q^2 = 0 \ . \tag{2.12}$$

Assuming that $L \neq 0$, we can pull out a common factor of L^2, leaving us with

$$r^2 - 3GMr + 2GQ^2 = 0 \ . \tag{2.13}$$

This is a simple quadratic equation, the roots of which are

$$r_c^{(\pm)} = \frac{1}{2}\left(3GM \pm \sqrt{9G^2M^2 - 8GQ^2}\right) \ . \tag{2.14}$$

Note that the extremality bound on the black hole requires $GQ^2 \leq G^2M^2$, and so the term in the square root is always positive, and thus $r_\pm$ are always real numbers. However, this is not enough; in order for the circular orbit to be allowed, it must fall outside of the black hole horizon! The outer horizon of a Reissner-Nordström black hole is located at

$$r_+ = GM + \sqrt{G^2M^2 - GQ^2} \ . \tag{2.15}$$

In order to compare the roots $r_c^{(\pm)}$ to the outer horizon r_+, let's first define

$$x \equiv \frac{Q^2}{GM^2} \ . \tag{2.16}$$

The extremality bound tells us that $x \in [0,1]$, with $x = 0$ corresponding to an uncharged Schwarzschild black hole and $x = 1$ corresponding to an extremal Reissner-Nordström black hole. This quantity is useful to define because we can express ratios of $r_c^{(\pm)}$ to r_+ purely in terms of x:

$$\frac{r_c^{(\pm)}}{r_+} = \frac{3 \pm \sqrt{9-8x}}{2 + 2\sqrt{1-x}} \ . \tag{2.17}$$

If you plot these functions on the interval $x \in [0,1]$, you will immediately see that $r_c^+ > r_+$ for all values of x, while $r_c^- \leq r_+$, with equality holding only when $x = 1$. Therefore, a generic Reissner-Nordström black hole has only one circular orbit, located at

$$r_c = \frac{1}{2}\left(3GM + \sqrt{9G^2M^2 - 8GQ^2}\right) \ . \tag{2.18}$$

If you plot out the potential, you should be able to see that this is an unstable circular orbit. The extremal Reissner-Nordström black hole has an additional second circular orbit right on the horizon, $r_c = r_+$. This is actually a stable orbit!

18.) In the Schwarzschild limit $Q = 0$, the photon sphere location goes to $r_c = 3GM$, which is precisely what we found when we analyzed the Schwarzschild geodesics previously. In the extremal limit, there are two circular orbits, one at $r_c = 2GM$ and one at $r_c = GM$. The first is the usual unstable photon sphere, while the second is a new stable orbit right at the horizon. This is a hint at something deep: extremal Reissner-Nordström black holes seem to have some special properties to them that generic black holes will not have. This is no accident; it has something to do with a concept known as *conformal symmetry.* I would love to chat with anyone who is interested about this.

19.) For a massive object, the circular orbit equation is a cubic polynomial. In principle it can be solved, since there is a closed-form expression for roots of cubic polynomials, but it is a cumbersome and difficult thing to work with. Moreover, it usually results in some imaginary roots, and so we would have to be very careful to analyze reality of roots, where they are in comparison to the horizon, etc. This takes a lot of work, and I didn't feel like making everyone suffer through it. But, hopefully by this point you should at least conceptually understand the steps you would have to take to do this analysis!